Cross-intersecting sub-families of hereditary families

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Abstract

Families $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k$ of sets are said to be *cross-intersecting* if for any i and j in $\{1, 2, \dots, k\}$ with $i \neq j$, any set in $\mathcal{A}_i$ intersects any set in $\mathcal{A}_j$. For a finite set X, let 2^X denote the *power set of* X (the family of all subsets of X). A family $\mathcal{H}$ is said to be *hereditary* if all subsets of any set in $\mathcal{H}$ are in $\mathcal{H}$; so $\mathcal{H}$ is hereditary if and only if it is a union of power sets. We conjecture that for any non-empty hereditary sub-family $\mathcal{H} \neq \{\emptyset\}$ of 2^X and any $k \geq |X| + 1$, both the sum and the product of sizes of k cross-intersecting sub-families $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k$ (not necessarily distinct or non-empty) of $\mathcal{H}$ are maxima if $\mathcal{A}_1 = \mathcal{A}_2 = \cdots = \mathcal{A}_k = \mathcal{S}$ for some largest *star* $\mathcal{S}$ *of* $\mathcal{H}$ (a sub-family of $\mathcal{H}$ whose sets have a common element). We prove this for the case when $\mathcal{H}$ is *compressed with respect to an element* x *of* X, and for this purpose we establish new properties of the usual *compression operation*. As we will show, for the sum, the condition $k \geq |X| + 1$ is sharp. However, for the product, we actually conjecture that the configuration $\mathcal{A}_1 = \mathcal{A}_2 = \cdots = \mathcal{A}_k = \mathcal{S}$ is optimal for any hereditary $\mathcal{H}$ and any $k \geq 2$, and we prove this for a special case.

1 Basic definitions and notation

Unless otherwise stated, we shall use small letters such as x to denote elements of a set or non-negative integers or functions, capital letters such as X to denote sets, and calligraphic letters such as $\mathcal{F}$ to denote *families* (i.e. sets whose elements are sets themselves). It is to be assumed that sets and families are *finite*. We call a set A an *r-element set*, or simply an *r-set*, if its size $|A|$ is r (i.e. if it contains exactly r elements).

For any integer $n \geq 1$, the set $\{1, \dots, n\}$ of the first n positive integers is denoted by $[n]$. For a set X, the *power set of* X (i.e. $\{A \colon A \subseteq X\}$) is denoted by 2^X, and the family of all r-element subsets of X is denoted by $\binom{X}{r}$.

A family $\mathcal{H}$ is said to be a *hereditary family* (also called an *ideal* or a *downset*) if all the subsets of any set in $\mathcal{H}$ are in $\mathcal{H}$. Clearly a family is hereditary if and only if it is a union of power sets. A *base of* $\mathcal{H}$ is a set in $\mathcal{H}$ that is not a subset of any other set in $\mathcal{H}$.

So a hereditary family is the union of power sets of its bases. An interesting example of a hereditary family is the family of all independent sets of a graph or matroid.

We will denote the union of all sets in a family $\mathcal{F}$ by $U(\mathcal{F})$. If x is an element of a set X, then we denote the family of those sets in $\mathcal{F}$ which contain x by $\mathcal{F}\langle x\rangle$, and we call $\mathcal{F}\langle x\rangle$ a *star of* $\mathcal{F}$. So $\mathcal{F}\langle x\rangle$ is the empty set $\emptyset$ if and only if x is not in $U(\mathcal{F})$.

A family $\mathcal{A}$ is said to be *intersecting* if any two sets in $\mathcal{A}$ intersect (i.e. contain at least one common element). We call a family $\mathcal{A}$ *centred* if the sets in $\mathcal{A}$ have a common element x (i.e. $\mathcal{A} = \mathcal{A}\langle x\rangle$). So a centred family is intersecting, and a non-empty star of a family $\mathcal{F}$ is centred. The simplest example of a non-centred intersecting family is the triangle $\{\{1,2\},\{1,3\},\{2,3\}\}$ (i.e. $\binom{[3]}{2}$).

For any family $\mathcal{A}$, let $\mathcal{A}^*$ denote the sub-family of $\mathcal{A}$ consisting of those sets in $\mathcal{A}$ that intersect each set in $\mathcal{A}$ (i.e. $\mathcal{A}^* = \{A \in \mathcal{A} \colon A \cap B \neq \emptyset \text{ for any } B \in \mathcal{A}\}$), and let $\mathcal{A}' = \mathcal{A} \backslash \mathcal{A}^*$. So $\mathcal{A}'$ consists of those sets in $\mathcal{A}$ that do not intersect all the sets in $\mathcal{A}$, and $\mathcal{A}^*$ is an intersecting family.

Families $\mathcal{A}_1, \dots, \mathcal{A}_k$ are said to be *cross-intersecting* if for any i and j in $[k]$ with $i \neq j$, any set in $\mathcal{A}_i$ intersects any set in $\mathcal{A}_j$.

If $U(\mathcal{F})$ has an element x such that $\mathcal{F}\langle x\rangle$ is a largest intersecting sub-family of $\mathcal{F}$ (i.e. no intersecting sub-family of $\mathcal{F}$ has more sets than $\mathcal{F}\langle x\rangle$), then we say that $\mathcal{F}$ has the *star property at* x. We simply say that $\mathcal{F}$ has the *star property* if either $U(\mathcal{F}) = \emptyset$ or $\mathcal{F}$ has the star property at some element of $U(\mathcal{F})$. For example, as was shown in [13], $2^{[n]}$ has the star property because, if $\mathcal{A}$ is an intersecting sub-family, then a subset A of $[n]$ and its complement $[n]\backslash A$ cannot both be in $\mathcal{A}$, and hence $|\mathcal{A}|$ is at most $\frac{1}{2}|2^{[n]}| = 2^{n-1}$, which is the size of the star $\{A \in 2^{[n]} \colon 1 \in A\}$. It may be that not all the largest intersecting sub-families of a family having the star property are stars; for example, the non-centred family $\{A \in 2^{[n]} \colon |A \cap [3]| \geq 2\}$ is an intersecting sub-family of $2^{[n]}$ of maximum size 2^{n-1}.

If $U(\mathcal{F})$ has an element x such that $(F\backslash\{y\}) \cup \{x\} \in \mathcal{F}$ whenever $y \in F \in \mathcal{F}$ and $x \notin F$, then $\mathcal{F}$ is said to be *compressed with respect to* x. For instance, this is the case when $\mathcal{F}$ is a hereditary family whose bases have a common element x (an interesting example is when $\mathcal{F}$ is the family of all independent sets of a graph that has an isolated vertex x).

A family $\mathcal{F} \subseteq 2^{[n]}$ is said to be *left-compressed* if $(F\backslash\{j\}) \cup \{i\} \in \mathcal{F}$ whenever $1 \leq i < j \in F \in \mathcal{F}$ and $i \notin F$.

2 Intersecting sub-families of hereditary families

The following is a famous longstanding open conjecture in extremal set theory due to Chvátal (see [7] for a more general conjecture).

Conjecture 2.1 ([11]) *If $\mathcal{H}$ is a hereditary family, then $\mathcal{H}$ has the star property.*

This conjecture was verified for the case when $\mathcal{H}$ is left-compressed by Chvátal [12] himself. Snevily [18] took this result (together with results in [17, 19]) a significant step forward by verifying Conjecture 2.1 for the case when $\mathcal{H}$ is compressed with respect to an element x of $U(\mathcal{H})$.

Theorem 2.2 ([18]) *If a hereditary family $\mathcal{H}$ is compressed with respect to an element x of $U(\mathcal{H})$, then $\mathcal{H}$ has the star property at x.*

A generalisation is proved in [7] by means of a self-contained alternative argument.

Snevily's proof of Theorem 2.2 makes use of the following interesting result of Berge [2] (a proof of which is also provided in [1, Chapter 6]).

Theorem 2.3 ([2]) *If $\mathcal{H}$ is a hereditary family, then $\mathcal{H}$ is a disjoint union of pairs of disjoint sets, together with $\emptyset$ if $|\mathcal{H}|$ is odd.*

This result was also motivated by Conjecture 2.1 as it has the following immediate consequence.

Corollary 2.4 *If $\mathcal{A}$ is an intersecting sub-family of a hereditary family $\mathcal{H}$, then*

$$|\mathcal{A}| \leq \frac{1}{2}|\mathcal{H}|.$$

Proof. For any pair of disjoint sets, at most only one set can be in an intersecting family $\mathcal{A}$. By Theorem 2.3, the result follows. □

A special case of Theorem 2.2 is a result of Schönheim [17] which says that Conjecture 2.1 is true if the bases of $\mathcal{H}$ have a common element, and this follows immediately from Corollary 2.4 and the following fact.

Proposition 2.5 ([17]) *If the bases of a hereditary family $\mathcal{H}$ have a common element x, then*

$$|\mathcal{H}\langle x\rangle| = \frac{1}{2}|\mathcal{H}|.$$

Proof. Partition $\mathcal{H}$ into $\mathcal{A} = \mathcal{H}\langle x\rangle$ and $\mathcal{B} = \{B \in \mathcal{H} \colon x \notin B\}$. If $A \in \mathcal{A}$ then $A\backslash\{x\} \in \mathcal{B}$; so $|\mathcal{A}| \leq |\mathcal{B}|$. If $B \in \mathcal{B}$ then $B \subset C$ for some base C of $\mathcal{H}$, and hence $B \cup \{x\} \in \mathcal{A}$ since $x \in C$; so $|\mathcal{B}| \leq |\mathcal{A}|$. Thus $|\mathcal{A}| = |\mathcal{B}| = \frac{1}{2}|\mathcal{H}|$. □

We outline an alternative proof of Proposition 2.5, using induction on $|U(\mathcal{H})|$. We have $x \in U(\mathcal{H})$. If $U(\mathcal{H}) = \{x\}$ then the result is trivial, so suppose $U(\mathcal{H})$ has an element $y \neq x$. We can apply the induction hypothesis to the hereditary families $\mathcal{I} = \{H\backslash\{y\} \colon H \in \mathcal{H}\langle y\rangle\}$ and $\mathcal{J} = \{H \in \mathcal{H} \colon y \notin H\}$ to obtain $|\mathcal{I}\langle x\rangle| = \frac{1}{2}|\mathcal{I}|$ and $|\mathcal{J}\langle x\rangle| = \frac{1}{2}|\mathcal{J}|$. Clearly $|\mathcal{H}| = |\mathcal{I}| + |\mathcal{J}|$ and $|\mathcal{H}\langle x\rangle| = |\mathcal{I}\langle x\rangle| + |\mathcal{J}\langle x\rangle|$. Hence the result.

Many other results and problems have been inspired by Conjecture 2.1 or are related to it; see [10, 16, 21]. In particular, an analogue of this conjecture for intersecting sub-families of $\mathcal{H}$ whose sets are of prescribed sizes is proved in [9] for the case when the bases are sufficiently large.

3 Cross-intersecting sub-families of hereditary families

For intersecting sub-families of a given family $\mathcal{F}$, the natural question to ask is how large they can be. Conjecture 2.1 claims that when $\mathcal{F}$ is hereditary we need only check the non-empty stars of $\mathcal{F}$ (of which there are $|U(\mathcal{F})|$). For cross-intersecting families, two natural parameters arise: the sum and the product of sizes of the cross-intersecting families (note that the product of sizes of k families $\mathcal{A}_1, \dots, \mathcal{A}_k$ is the number of k-tuples $(A_1, \dots, A_k)$ such that $A_i \in \mathcal{A}_i$ for each $i \in [k]$). It is therefore natural to consider the problem of maximising the sum or the product of sizes of k cross-intersecting sub-families (not necessarily distinct or non-empty) of a given family $\mathcal{F}$ (see [8]). To the best of the author's knowledge, the first time a problem of this kind was considered was in [15], which gives the solution to the sum problem for $\mathcal{F} = \binom{[n]}{r}$. We suggest a few conjectures for the case when $\mathcal{F}$ is hereditary, and we prove that they are true in some important cases. Obviously, any family $\mathcal{F}$ is a sub-family of 2^X with $X = U(\mathcal{F})$, and we may assume that $X = [n]$.

For the sum of sizes, we suggest the following.

Conjecture 3.1 *If $k \geq n+1$ and $\mathcal{A}_1, \dots, \mathcal{A}_k$ are cross-intersecting sub-families of a hereditary sub-family $\mathcal{H} \neq \{\emptyset\}$ of $2^{[n]}$, then the sum $\sum_{i=1}^{k} |\mathcal{A}_i|$ is maximum if $\mathcal{A}_1 = \dots = \mathcal{A}_k = \mathcal{S}$ for some largest star $\mathcal{S}$ of $\mathcal{H}$.*

The condition that $k \geq n+1$ is sharp. Indeed, consider $\mathcal{H} = \{\emptyset\} \cup \binom{[n]}{1}$ and $2 \leq k < n+1$. Let $\mathcal{S} = \{\{1\}\}$; so $\mathcal{S}$ is a largest star of $\mathcal{H}$. Let $\mathcal{A}_1 = \dots = \mathcal{A}_k = \mathcal{S}$, and let $\mathcal{B}_1 = \mathcal{H}$ and $\mathcal{B}_2 = \dots = \mathcal{B}_k = \emptyset$. Then $\mathcal{A}_1, \dots, \mathcal{A}_k$ are cross-intersecting, $\mathcal{B}_1, \dots, \mathcal{B}_k$ are cross-intersecting, and $\sum_{i=1}^{k} |\mathcal{A}_i| = k < n+1 = \sum_{i=1}^{k} |\mathcal{B}_i|$. Also, we cannot remove the condition that $\mathcal{H} \neq \{\emptyset\}$. Indeed, suppose $\mathcal{H} = \{\emptyset\}$; so $\mathcal{S} = \emptyset$ is the only star of $\mathcal{H}$. Thus, if $\mathcal{A}_1 = \dots = \mathcal{A}_k = \mathcal{S}$, $\mathcal{B}_1 = \mathcal{H}$ and $\mathcal{B}_2 = \dots = \mathcal{B}_k = \emptyset$, then $\sum_{i=1}^{k} |\mathcal{A}_i| = 0 < 1 = \sum_{i=1}^{k} |\mathcal{B}_i|$.

For the general case when we have any number of cross-intersecting families, we suggest the following stronger conjecture.

Conjecture 3.2 *Let $\mathcal{A}_1, \dots, \mathcal{A}_k$ be cross-intersecting sub-families of a non-empty hereditary sub-family $\mathcal{H} \neq \{\emptyset\}$ of $2^{[n]}$, and let $\mathcal{S}$ be a largest star of $\mathcal{H}$.*
(i) If $k \leq \frac{|\mathcal{H}|}{|\mathcal{S}|}$, then $\sum_{i=1}^{k} |\mathcal{A}_i|$ is maximum if $\mathcal{A}_1 = \mathcal{H}$ and $\mathcal{A}_2 = \dots = \mathcal{A}_k = \emptyset$.
(ii) If $k \geq \frac{|\mathcal{H}|}{|\mathcal{S}|}$, then $\sum_{i=1}^{k} |\mathcal{A}_i|$ is maximum if $\mathcal{A}_1 = \dots = \mathcal{A}_k = \mathcal{S}$.

This conjecture is simply saying that at least one of the two simple configurations $\mathcal{A}_1 = \mathcal{H}$, $\mathcal{A}_2 = \dots = \mathcal{A}_k = \emptyset$ and $\mathcal{A}_1 = \dots = \mathcal{A}_k = \mathcal{S}$ gives a maximum sum of sizes. This strengthens Conjecture 3.1 because, since $\mathcal{H}$ has a non-empty set (as $\mathcal{H} \neq \emptyset$ and $\mathcal{H} \neq \{\emptyset\}$), we have $\mathcal{S} \neq \emptyset$, $|\mathcal{H}| = |\{\emptyset\} \cup \bigcup_{i=1}^{n} \mathcal{H}\langle i \rangle| \leq 1 + n|\mathcal{S}| \leq (n+1)|\mathcal{S}|$ and hence $\frac{|\mathcal{H}|}{|\mathcal{S}|} \leq n+1$; that is, if (ii) is true, then Conjecture 3.1 follows.

For the product of sizes, we first present the following consequence of Conjecture 3.1.

Proposition 3.3 *Let $A_1, \dots, \mathcal{A}_k, \mathcal{H}$ and $\mathcal{S}$ be as in Conjecture 3.1. If Conjecture 3.1 is true, then the product $\prod_{i=1}^{k} |\mathcal{A}_i|$ is maximum if $\mathcal{A}_1 = \dots = \mathcal{A}_k = \mathcal{S}$.*

This follows immediately from the following elementary result, known as the Arithmetic Mean-Geometric Mean (AM-GM) Inequality.

Lemma 3.4 (AM-GM Inequality) *If $x_1, x_2, \dots, x_k$ are non-negative real numbers, then*

$$\left(\prod_{i=1}^{k} x_i\right)^{1/k} \leq \frac{1}{k}\sum_{i=1}^{k} x_i.$$

Indeed, suppose Conjecture 3.1 is true. Then $\sum_{i=1}^{k} |\mathcal{A}_i| \leq k|\mathcal{S}|$. Thus, by Lemma 3.4, $\left(\prod_{i=1}^{k} |\mathcal{A}_i|\right)^{1/k} \leq |\mathcal{S}|$ and hence Proposition 3.3.

However, we conjecture the following stronger statement about the maximum product.

Conjecture 3.5 *If $k \geq 2$ and $\mathcal{A}_1, \dots, \mathcal{A}_k$ are cross-intersecting sub-families of a hereditary family $\mathcal{H}$, then $\prod_{i=1}^{k} |\mathcal{A}_i|$ is maximum if $\mathcal{A}_1 = \dots = \mathcal{A}_k = \mathcal{S}$ for some largest star $\mathcal{S}$ of $\mathcal{H}$.*

Proposition 3.6 *Conjecture 3.5 is true if it is true for $k = 2$.*

Proof. Let $h \geq 2$, and let $\mathcal{A}_1, \dots, \mathcal{A}_h$ be cross-intersecting sub-families of a hereditary family $\mathcal{H}$. For each $i \in [h]$, let $a_i = |\mathcal{A}_i|$. Let s be the size of a largest star of $\mathcal{H}$. Suppose Conjecture 3.5 is true for $k = 2$. Then $a_i a_j \leq s^2$ for any $i, j \in [h]$ with $i \neq j$. Let mod* represent the usual *modulo operation* with the exception that for any two integers x and y, $(xy) \bmod^* y$ is y instead of 0. We have

$$\left(\prod_{i=1}^{h} a_i\right)^2 = (a_1 a_2)(a_{3 \bmod^* h} a_{4 \bmod^* h}) \cdots (a_{(2h-1) \bmod^* h} a_{(2h) \bmod^* h}) \leq (s^2)^h = (s^h)^2.$$

So $\prod_{i=1}^{h} a_i \leq s^h$. Hence the result. □

Each of the above conjectures generalises Conjecture 2.1. Indeed, let $\mathcal{A}$ be an intersecting sub-family of a hereditary family $\mathcal{H} \subseteq 2^{[n]}$ with $U(\mathcal{H}) \neq \emptyset$, and let $\mathcal{S}$ be a largest star of $\mathcal{H}$. Let $k \geq n+1$, and let $\mathcal{A}_1 = \dots = \mathcal{A}_k = \mathcal{A}$. Then $\mathcal{A}_1, \dots, \mathcal{A}_k$ are cross-intersecting. Thus, each of Conjectures 3.1, 3.2 and 3.5 claims that $|\mathcal{A}_i| \leq |\mathcal{S}|$ for each $i \in [k]$ (since $\mathcal{A}_1 = \dots = \mathcal{A}_k$), and hence $|\mathcal{A}| \leq |\mathcal{S}|$ as claimed by Conjecture 2.1.

All the above conjectures are true for the special case when $\mathcal{H} = 2^{[n]}$; more precisely, the following holds.

Theorem 3.7 (see [8]) *For any $k \geq 2$, both the sum and the product of sizes of k cross-intersecting sub-families $\mathcal{A}_1, \dots, \mathcal{A}_k$ of $2^{[n]}$ are maxima if $\mathcal{A}_1 = \dots = \mathcal{A}_k = \{A \subseteq [n] \colon 1 \in A\}$.*

We generalise this result as follows.

Theorem 3.8 *If $\mathcal{A}_1, \dots, \mathcal{A}_k$ are cross-intersecting sub-families of a hereditary family $\mathcal{H}$, then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq k\frac{|\mathcal{H}|}{2} \quad \textit{and} \quad \prod_{i=1}^{k} |\mathcal{A}_i| \leq \left(\frac{|\mathcal{H}|}{2}\right)^k.$$

Moreover, both bounds are attained if the bases of $\mathcal{H}$ have a common element x and $\mathcal{A}_1 = \dots = \mathcal{A}_k = \mathcal{H}\langle x \rangle$.

Proof. Theorem 2.3 tells us that there exists a partition $\mathcal{H}_1 \cup \mathcal{H}_2 \cup \dots \cup \mathcal{H}_m$ of $\mathcal{H}$ such that $m = \left\lceil \frac{|\mathcal{H}|}{2} \right\rceil$, $\mathcal{H}_i = \{H_{i,1}, H_{i,2}\}$ for some $H_{i,1}, H_{i,2} \in \mathcal{H}$ with $H_{i,1} \cap H_{i,2} = \emptyset$, $i = 1, \dots, m$, and if $|\mathcal{H}|$ is odd then $H_{m,1} = H_{m,2} = \emptyset$.

Let $\mathcal{A} = \bigcup_{i=1}^k \mathcal{A}_i$. By the cross-intersection condition, we clearly have $\mathcal{A}^* = \bigcup_{i=1}^k \mathcal{A}_i^*$ and $\mathcal{A}' = \bigcup_{i=1}^k \mathcal{A}_i'$. Suppose $\mathcal{A}_i' \cap \mathcal{A}_j' \neq \emptyset$ for some $i \neq j$. Let $A \in \mathcal{A}_i' \cap \mathcal{A}_j'$. Then there exists $A_i \in \mathcal{A}_i$ such that $A \cap A_i = \emptyset$, but this is a contradiction because $A \in \mathcal{A}_j$. So $\mathcal{A}_i' \cap \mathcal{A}_j' = \emptyset$ for any $i \neq j$. Therefore $|\mathcal{A}'| = \sum_{i=1}^k |\mathcal{A}_i'|$.

Let $\mathcal{B} = \{H_{i,j} \colon i \in [m], j \in [2], H_{i,3-j} \in \mathcal{A}^*\}$. So $|\mathcal{B}| = |\mathcal{A}^*|$. For any $H_{i,j} \in \mathcal{B}$, $H_{i,j} \notin \mathcal{A}$ since $H_{i,3-j} \in \mathcal{A}^*$ and $H_{i,j} \cap H_{i,3-j} = \emptyset$. So $\mathcal{A}$ and $\mathcal{B}$ are disjoint sub-families of $\mathcal{H}$. Therefore,

$$2|\mathcal{A}^*| + |\mathcal{A}'| = |\mathcal{A}^*| + |\mathcal{B}| + |\mathcal{A}'| = |\mathcal{A}| + |\mathcal{B}| = |\mathcal{A} \cup \mathcal{B}| \leq |\mathcal{H}|$$

and hence, dividing throughout by 2, we get $|\mathcal{A}^*| + \frac{1}{2}|\mathcal{A}'| \leq \frac{1}{2}|\mathcal{H}|$. So we have

$$\sum_{i=1}^{k} |\mathcal{A}_i| = \sum_{i=1}^{k} |\mathcal{A}_i'| + \sum_{i=1}^{k} |\mathcal{A}_i^*| \leq |\mathcal{A}'| + k|\mathcal{A}^*| \leq k\left(|\mathcal{A}^*| + \frac{1}{2}|\mathcal{A}'|\right) \leq k\frac{|\mathcal{H}|}{2}$$

and hence, by Lemma 3.4,

$$\prod_{i=1}^{k} |\mathcal{A}_i| \leq \left(\frac{1}{k}\sum_{i=1}^{k} |\mathcal{A}_i|\right)^k \leq \left(\frac{|\mathcal{H}|}{2}\right)^k.$$

The second part of the theorem is an immediate consequence of Proposition 2.5. □

Note that it is immediate from the above proof that if $k \geq 3$, then each of the two bounds is attained only if $\mathcal{A}_1 = \dots = \mathcal{A}_k = \mathcal{A}^* = \mathcal{A}$ and $\mathcal{A}$ is a largest intersecting sub-family of $\mathcal{H}$.

Corollary 3.9 *Conjectures 3.1, 3.2 and 3.5 are true if the bases of $\mathcal{H}$ have a common element.*

Proof. If the bases of $\mathcal{H}$ have a common element x, then by Corollary 2.4 and Proposition 2.5, $\mathcal{H}\langle x \rangle$ is a largest star of $\mathcal{H}$ of size $\frac{1}{2}|\mathcal{H}|$. By Theorem 3.8, the result follows. □

Corollary 3.10 *Conjecture 3.2 is true if* $k = 2$.

Proof. By Corollary 2.4, we have $|\mathcal{S}| \leq \frac{1}{2}|\mathcal{H}|$ and hence $2 \leq \frac{|\mathcal{H}|}{|\mathcal{S}|}$. Now by Theorem 3.8, $|\mathcal{A}_1| + |\mathcal{A}_2| \leq |\mathcal{H}|$. Hence the result. □

We now come to our main result, which verifies Conjectures 3.1, 3.2 and 3.5 for the case when $k \geq n+1$ and $\mathcal{H}$ is compressed with respect to an element of $[n]$. As we remarked earlier, an important example of such a hereditary family is one whose bases have a common element. Other important examples include $\bigcup_{r=0}^{m} \binom{[n]}{r}$ for any $m \in \{0\} \cup [n]$ (for $m = n$ we get $2^{[n]}$).

Theorem 3.11 *Let* $\mathcal{H}$ *be a hereditary sub-family of* $2^{[n]}$ *that is compressed with respect to an element* x *of* $[n]$, *and let* $\mathcal{S} = \mathcal{H}\langle x \rangle$. *Let* $k \geq n+1$, *and let* $\mathcal{A}_1, \dots, \mathcal{A}_k$ *be cross-intersecting sub-families of* $\mathcal{H}$. *Then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq k|\mathcal{S}| \quad \text{and} \quad \prod_{i=1}^{k} |\mathcal{A}_i| \leq |\mathcal{S}|^k,$$

and both bounds are attained if $\mathcal{A}_1 = \cdots = \mathcal{A}_k = \mathcal{S}$. *Moreover:*
(a) $\sum_{i=1}^{k} |\mathcal{A}_i| = k|\mathcal{S}|$ *if and only if either* $\mathcal{A}_1 = \cdots = \mathcal{A}_k = \mathcal{L}$ *for some largest intersecting sub-family* $\mathcal{L}$ *of* $\mathcal{H}$ *or* $k = n+1$ *and for some* $i \in [k]$, $\mathcal{A}_i = \mathcal{H} = \{\emptyset\} \cup \binom{[n]}{1}$ *and* $\mathcal{A}_j = \emptyset$ *for each* $j \in [k]\backslash\{i\}$.
(b) $\prod_{i=1}^{k} |\mathcal{A}_i| = |\mathcal{S}|^k$ *if and only if* $\mathcal{A}_1 = \cdots = \mathcal{A}_k = \mathcal{L}$ *for some largest intersecting sub-family* $\mathcal{L}$ *of* $\mathcal{H}$.

This generalises Theorem 2.2 in the same way that Conjectures 3.1, 3.2 and 3.5 generalise Conjecture 2.1 (as explained above). We prove this result in Section 5; however, we set up the necessary tools in the next section.

4 New properties of the compression operation

The proof of Theorem 3.11 will be based on the compression technique, which featured in the original proof of the classical Erdős-Ko-Rado Theorem [13].

For a non-empty set X and $x, y \in X$, let $\delta_{x,y} \colon 2^X \to 2^X$ be defined by

$$\delta_{x,y}(A) = \begin{cases} (A\backslash\{y\}) \cup \{x\} & \text{if } y \in A \text{ and } x \notin A; \\ A & \text{otherwise,} \end{cases}$$

and let $\Delta_{x,y} \colon 2^{2^X} \to 2^{2^X}$ be the *compression operation* (see [13]) defined by

$$\Delta_{x,y}(\mathcal{A}) = \{\delta_{x,y}(A) \colon A \in \mathcal{A}, \delta_{x,y}(A) \notin \mathcal{A}\} \cup \{A \in \mathcal{A} \colon \delta_{x,y}(A) \in \mathcal{A}\}.$$

Note that $|\Delta_{x,y}(\mathcal{A})| = |\mathcal{A}|$. It is well-known, and easy to check, that $\Delta_{x,y}(\mathcal{A})$ is intersecting if $\mathcal{A}$ is intersecting; [14] provides a survey on the properties and uses of compression (also called *shifting*) operations in extremal set theory. We now establish new properties of compressions for the purpose of proving Theorem 3.11. Recall the definition of $\mathcal{A}^*$ and $\mathcal{A}'$ in Section 1. We will mainly (but not solely) prove that a compression on a family $\mathcal{A}$ can only increase the size of $\mathcal{A}^*$ (i.e. the number of sets that intersect every other set).

Lemma 4.1 *Let $\mathcal{A}$ be a sub-family of $2^{[n]}$ with $n \geq 2$, and let $\mathcal{B} = \Delta_{i,j}(\mathcal{A})$ for some $i, j \in [n]$ with $i \neq j$. Then:*
(i) if $A \in \mathcal{A}^*$, then $\delta_{i,j}(A) \in \mathcal{B}^*$;
(ii) if $A \in \mathcal{A}^* \backslash \mathcal{B}^*$, then $\delta_{i,j}(A) \notin \mathcal{A}^*$;
(iii) if $B \in \mathcal{B}^*$, then $\delta_{i,j}(B) \in \mathcal{B}^*$;
(iv) $|\mathcal{A}^*| \leq |\mathcal{B}^*|$.

Proof. Suppose $A \in \mathcal{A}^*$. Obviously $\delta_{i,j}(A) \in \mathcal{B}$. If $i \in A$, then $\delta_{i,j}(A) \in \mathcal{B}^*$ because $\delta_{i,j}(A) = A$, $i \in B$ for any $B \in \mathcal{B} \backslash \mathcal{A}$, and $A \cap B' \neq \emptyset$ for any $B' \in \mathcal{A}$ (since $A \in \mathcal{A}^*$). Suppose $i \notin A$. If $j \notin A$, then $(A \cap A') \backslash \{i, j\} = A \cap A' \neq \emptyset$ for any $A' \in \mathcal{A}$, and hence $\delta_{i,j}(A) \in \mathcal{B}^*$. Suppose $j \in A$. Then $i \in \delta_{i,j}(A) \neq A$. Suppose $\delta_{i,j}(A) \notin \mathcal{B}^*$. Then $\delta_{i,j}(A) \cap C = \emptyset$ for some $C \in \mathcal{B}$. So $i \notin C$ and hence $C \in \mathcal{A}$. So $C \in \mathcal{A} \cap \mathcal{B}$ and hence $C, \delta_{i,j}(C) \in \mathcal{A}$. So A intersects both C and $\delta_{i,j}(C)$. From $\delta_{i,j}(A) \cap C = \emptyset$ and $A \cap C \neq \emptyset$ we get $A \cap C = \{j\}$. But this yields the contradiction that $A \cap \delta_{i,j}(C) = \emptyset$. Hence (i).

Suppose $A \in \mathcal{A}^* \backslash \mathcal{B}^*$. If $\delta_{i,j}(A) \notin \mathcal{A}$, then obviously $\delta_{i,j}(A) \notin \mathcal{A}^*$. Now suppose $\delta_{i,j}(A) \in \mathcal{A}$. Then $A \in \mathcal{B}$. Since $A \notin \mathcal{B}^*$, $A \cap D = \emptyset$ for some $D \in \mathcal{B}$. Since A intersects each set in $\mathcal{A}$, we must have $D = \delta_{i,j}(E) \neq E$ for some $E \in \mathcal{A}$ with $A \cap E = \{j\}$ and $i \notin A \cup E$. Thus $\delta_{i,j}(A) \cap E = \emptyset$. So $\delta_{i,j}(A) \notin \mathcal{A}^*$. Hence (ii).

Suppose $B \in \mathcal{B}^*$. If $\delta_{i,j}(B) = B$, then obviously $\delta_{i,j}(B) \in \mathcal{B}^*$. Suppose $\delta_{i,j}(B) \neq B$. Then $B, \delta_{i,j}(B) \in \mathcal{A}$. Thus, since B intersects every set in $\mathcal{B}$ and $i \notin B$, B intersects every set in $\mathcal{A}$, and hence $B \in \mathcal{A}^*$. By (i), $\delta_{i,j}(B) \in \mathcal{B}^*$. Hence (iii).

By (i), we can define a function $f : \mathcal{A}^* \to \mathcal{B}^*$ by

$$f(A) = \begin{cases} A & \text{if } A \in \mathcal{A}^* \cap \mathcal{B}^*; \\ \delta_{i,j}(A) & \text{if } A \in \mathcal{A}^* \backslash \mathcal{B}^*. \end{cases}$$

Suppose $A_1, A_2 \in \mathcal{A}^*$ such that $f(A_1) = f(A_2)$. Suppose $A_1 \in \mathcal{A}^* \cap \mathcal{B}^*$ and $A_2 \in \mathcal{A}^* \backslash \mathcal{B}^*$; then we have $\delta_{i,j}(A_2) = f(A_2) = f(A_1) = A_1 \in \mathcal{A}^*$, which is a contradiction because $\delta_{i,j}(A_2) \notin \mathcal{A}^*$ by (ii). Similarly, we cannot have $A_2 \in \mathcal{A}^* \cap \mathcal{B}^*$ and $A_1 \in \mathcal{A}^* \backslash \mathcal{B}^*$. If $A_1, A_2 \in \mathcal{A}^* \cap \mathcal{B}^*$, then we have $A_1 = f(A_1) = f(A_2) = A_2$. Finally, suppose $A_1, A_2 \in \mathcal{A}^* \backslash \mathcal{B}^*$. Then we have $\delta_{i,j}(A_1) = f(A_1) = f(A_2) = \delta_{i,j}(A_2)$ and, by (ii), $\delta_{i,j}(A_1) \neq A_1$ and $\delta_{i,j}(A_2) \neq A_2$. So $A_1 = \delta_{j,i}(\delta_{i,j}(A_1)) = \delta_{j,i}(\delta_{i,j}(A_2)) = A_2$. Therefore, no two distinct sets in $\mathcal{A}^*$ are mapped by f to the same set in $\mathcal{B}^*$ (i.e. f is injective). Hence (iv). $\square$

5 Proof of Theorem 3.11

We now prove Theorem 3.11. We follow the strategy introduced in [3, 4] and also adopted in [5, 6, 8, 20], which mainly is to determine (or at least obtain a reasonable lower bound for), for the family $\mathcal{F}$ under consideration, the largest rational number $c \leq l/|\mathcal{F}|$ such that $|\mathcal{A}^*| + c|\mathcal{A}'| \leq l$ for any sub-family $\mathcal{A}$ of $\mathcal{F}$, where l is the size of a largest intersecting sub-family of $\mathcal{F}$; see [8] for a detailed general explanation. For this purpose we shall first prove the following result, and this will be the most technically complex part of proving Theorem 3.11.

Theorem 5.1 *Let $\mathcal{H}$ be a hereditary sub-family of $2^{[n]}$ that is compressed with respect to an element x of $[n]$, and let $\mathcal{A}$ be a sub-family of $\mathcal{H}$. Then*

$$|\mathcal{A}^*| + \frac{1}{n+1}|\mathcal{A}'| \leq |\mathcal{H}\langle x \rangle|,$$

and if $\mathcal{A}' \neq \emptyset$, then equality holds if and only if $\mathcal{A} = \mathcal{H} = \{\emptyset\} \cup \binom{[n]}{1}$.

Proof. Since $\mathcal{H}$ is compressed with respect to x, we have $x \in U(\mathcal{H})$ and hence $\mathcal{H}\langle x \rangle \neq \emptyset$. The result is trivial if $n = 1$, so we consider $n \geq 2$ and proceed by induction on n.

We may assume that $x = 1$. Let $\mathcal{L} = \mathcal{H}\langle 1 \rangle$. Let $\mathcal{B} = \Delta_{1,n}(\mathcal{A})$. Given that $\mathcal{H}$ is compressed with respect to 1, we have $\mathcal{B} \subseteq \mathcal{H}$. Define

$$\begin{aligned}
\mathcal{B}_1 &= \{B \in \mathcal{B} \colon n \in B\},\\
\mathcal{B}_2 &= \{B \backslash \{n\} \colon B \in \mathcal{B}_1\},\\
\mathcal{B}_3 &= \mathcal{B} \backslash \mathcal{B}_1 = \{B \in \mathcal{B} \colon n \notin B\}.
\end{aligned}$$

Define $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ and $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ similarly. So $\mathcal{B}_2, \mathcal{L}_2 \subseteq \mathcal{H}_2 \subseteq 2^{[n-1]}$ and $\mathcal{B}_3, \mathcal{L}_3 \subseteq \mathcal{H}_3 \subseteq 2^{[n-1]}$. Also note that the properties of $\mathcal{H}$ are inherited by $\mathcal{H}_3$, that is, $\mathcal{H}_3$ is hereditary and compressed with respect to 1; the same holds for $\mathcal{H}_2$ unless $U(\mathcal{H}_2) = \emptyset$ (in which case $\mathcal{H}_1$ is either $\emptyset$ or $\{\{n\}\}$). Define

$$\begin{aligned}
\mathcal{C}_1 &= \{C \in \mathcal{B}_1 \colon 1 \in C,\ B \cap C = \{n\} \text{ for some } B \in \mathcal{B}_1\},\\
\mathcal{C}_2 &= \{C \backslash \{n\} \colon C \in \mathcal{C}_1,\ C \backslash \{n\} \notin \mathcal{B}_3\},\\
\mathcal{D} &= \mathcal{B}_2 \backslash \mathcal{C}_2,\\
\mathcal{E} &= \mathcal{B}_3 \cup \mathcal{C}_2.
\end{aligned}$$

Obviously $\mathcal{C}_2 \subseteq \mathcal{B}_2$ and $\mathcal{D} \subseteq \mathcal{H}_2$. Given that $\mathcal{H}$ is hereditary, we clearly have $\mathcal{C}_2 \subseteq \mathcal{H}_3$; so $\mathcal{E} \subseteq \mathcal{H}_3$. Note that $\mathcal{L}_2 = \mathcal{H}_2\langle 1 \rangle$ and $\mathcal{L}_3 = \mathcal{H}_3\langle 1 \rangle$. Therefore, by the induction hypothesis, we have $|\mathcal{E}^*| + \frac{1}{n}|\mathcal{E}'| \leq |\mathcal{L}_3|$, and if $U(\mathcal{H}_2) \neq \emptyset$, then $|\mathcal{D}^*| + \frac{1}{n}|\mathcal{D}'| \leq |\mathcal{L}_2|$.

By definition of $\mathcal{C}_2$, we have $\mathcal{B}_3 \cap \mathcal{C}_2 = \emptyset$ and hence $|\mathcal{E}| = |\mathcal{B}_3| + |\mathcal{C}_2|$. Since $\mathcal{C}_2 \subseteq \mathcal{B}_2$, we have $|\mathcal{D}| = |\mathcal{B}_2| - |\mathcal{C}_2|$. So $|\mathcal{D}| + |\mathcal{E}| = |\mathcal{B}_2| + |\mathcal{B}_3|$ and hence, since $|\mathcal{D}| + |\mathcal{E}| = |\mathcal{D}^*| + |\mathcal{D}'| + |\mathcal{E}^*| + |\mathcal{E}'|$ and $|\mathcal{B}_2| + |\mathcal{B}_3| = |\mathcal{B}| = |\mathcal{B}^*| + |\mathcal{B}'|$,

$$|\mathcal{D}^*| + |\mathcal{E}^*| + |\mathcal{D}'| + |\mathcal{E}'| = |\mathcal{B}^*| + |\mathcal{B}'|. \tag{1}$$

We now come to our main step, which is to show that $|\mathcal{B}^*| \leq |\mathcal{D}^*| + |\mathcal{E}^*|$. So suppose $\mathcal{B}^*$ contains a set B.

First, suppose $n \notin B$. Then clearly B intersects all sets in $\mathcal{B}_2 \cup \mathcal{B}_3$ and hence, since $\mathcal{C}_2 \subseteq \mathcal{B}_2$, we have $B \in \mathcal{E}^*$. Also, $B \notin \mathcal{C}_2$ since $B \in \mathcal{B}_3$. In brief, we have

$$n \notin B \in \mathcal{B}^* \quad \Rightarrow \quad B \in (\mathcal{E}^* \backslash \mathcal{C}_2) \cap \mathcal{B}^*. \tag{2}$$

Now suppose $n \in B$, that is, $B \in \mathcal{B}_1$. Let $B^- = B \backslash \{n\}$. Clearly B^- intersects all sets in $\mathcal{B}_3$. If $B^- \in \mathcal{C}_2$ then, since all sets in $\mathcal{C}_2$ contain 1, B^- also intersects each set in $\mathcal{C}_2$, meaning that $B^- \in \mathcal{E}^*$. In brief, we have

$$n \in B \in \mathcal{B}^*,\ B \backslash \{n\} \in \mathcal{C}_2 \quad \Rightarrow \quad B \backslash \{n\} \in \mathcal{E}^* \cap \mathcal{C}_2. \tag{3}$$

Suppose $B^- \notin \mathcal{C}_2$. Then $B^- \in \mathcal{D}$. Suppose $B^- \notin \mathcal{D}^*$. Then $B^- \cap D = \emptyset$ for some $D \in \mathcal{D}$, and hence, setting $D^+ = D \cup \{n\}$, we have $B \cap D^+ = \{n\}$ and $D^+ \in \mathcal{B}_1$. Since $B \cap D = \emptyset$, D cannot be in $\mathcal{B}_3$. Thus we must have $1 \notin D^+$, because otherwise we get $D^+ \in \mathcal{C}_1$ and hence $D \in \mathcal{C}_2$ (contradicting $D \in \mathcal{D}$). It follows that we must also have $1 \in B$, because otherwise we get $\delta_{1,n}(B) \cap D^+ = \emptyset$, contradicting Lemma 4.1(iii). So $B \in \mathcal{C}_1$. Thus, since $B^- \notin \mathcal{C}_2$, B^- must be in $\mathcal{B}_3$ and hence in $\mathcal{B}$. Since $B \cap D^+ = \{n\}$, we have $B^- \cap D^+ = \emptyset$ and hence $B^- \notin \mathcal{B}^*$. However, since B^- intersects all sets in $\mathcal{B}_3$ and $1 \in B^- \cap C$ for any $C \in \mathcal{C}_2$, we have $B^- \in \mathcal{E}^*$. So we have just shown that

$$n \in B \in \mathcal{B}^*,\ B \backslash \{n\} \notin \mathcal{C}_2,\ B \backslash \{n\} \notin \mathcal{D}^* \quad \Rightarrow \quad B \backslash \{n\} \in \mathcal{E}^* \backslash (\mathcal{C}_2 \cup \mathcal{B}^*). \tag{4}$$

Define

$$\begin{aligned}
\mathcal{F}_1 &= \{F \in \mathcal{B}^* \colon n \notin F\},\\
\mathcal{F}_2 &= \{F \in \mathcal{B}^* \colon n \in F, F \backslash \{n\} \in \mathcal{C}_2\},\\
\mathcal{F}_3 &= \{F \in \mathcal{B}^* \colon n \in F, F \backslash \{n\} \notin \mathcal{C}_2, F \backslash \{n\} \notin \mathcal{D}^*\},\\
\mathcal{F}_4 &= \{F \in \mathcal{B}^* \colon n \in F, F \backslash \{n\} \notin \mathcal{C}_2, F \backslash \{n\} \in \mathcal{D}^*\}.
\end{aligned}$$

Clearly $|\mathcal{B}^*| = |\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3| + |\mathcal{F}_4|$ and $|\mathcal{F}_4| \leq |\mathcal{D}^*|$. Also, by (2) - (4), we have $|\mathcal{F}_1| \leq |(\mathcal{E}^* \backslash \mathcal{C}_2) \cap \mathcal{B}^*|$, $|\mathcal{F}_2| \leq |\mathcal{E}^* \cap \mathcal{C}_2|$ and $|\mathcal{F}_3| \leq |\mathcal{E}^* \backslash (\mathcal{C}_2 \cup \mathcal{B}^*)|$. Thus, since $(\mathcal{E}^* \backslash \mathcal{C}_2) \cap \mathcal{B}^*$, $\mathcal{E}^* \cap \mathcal{C}_2$ and $\mathcal{E}^* \backslash (\mathcal{C}_2 \cup \mathcal{B}^*)$ are disjoint sub-families of $\mathcal{E}^*$, we obtain $|\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3| \leq |\mathcal{E}^*|$. So $|\mathcal{B}^*| \leq |\mathcal{D}^*| + |\mathcal{E}^*|$ as required.

We now know that $|\mathcal{D}^*| + |\mathcal{E}^*| = |\mathcal{B}^*| + p$ for some integer $p \geq 0$. By (1), we therefore have $|\mathcal{D}'| + |\mathcal{E}'| = (|\mathcal{B}^*| + |\mathcal{B}'|) - (|\mathcal{B}^*| + p) = |\mathcal{B}'| - p$.

At this point, we need to divide the problem into two cases.

Case 1: $U(\mathcal{H}_2) \neq \emptyset$. So $|\mathcal{D}^*| + \frac{1}{n}|\mathcal{D}'| \leq |\mathcal{L}_2|$. Since we earlier obtained $|\mathcal{E}^*| + \frac{1}{n}|\mathcal{E}'| \leq |\mathcal{L}_3|$,

$$|\mathcal{D}^*| + |\mathcal{E}^*| + \frac{1}{n}(|\mathcal{D}'| + |\mathcal{E}'|) \leq |\mathcal{L}_2| + |\mathcal{L}_3| = |\mathcal{L}_1| + |\mathcal{L}_3| = |\mathcal{L}|.$$

We now have

$$|\mathcal{B}^*| + \frac{1}{n}|\mathcal{B}'| \leq |\mathcal{B}^*| + p + \frac{1}{n}(|\mathcal{B}'| - p) = |\mathcal{D}^*| + |\mathcal{E}^*| + \frac{1}{n}(|\mathcal{D}'| + |\mathcal{E}'|) \leq |\mathcal{L}|.$$

Since $|\mathcal{A}^*|+|\mathcal{A}'| = |\mathcal{A}| = |\mathcal{B}| = |\mathcal{B}^*|+|\mathcal{B}'|$, Lemma 4.1(iv) gives us $|\mathcal{A}^*|+\frac{1}{n}|\mathcal{A}'| \leq |\mathcal{B}^*|+\frac{1}{n}|\mathcal{B}'|$. So $|\mathcal{A}^*| + \frac{1}{n+1}|\mathcal{A}'| \leq |\mathcal{L}|$, and the inequality is strict if $\mathcal{A}' \neq \emptyset$.

Case 2: $U(\mathcal{H}_2) = \emptyset$. So $\mathcal{H}_1$ is either $\emptyset$ or $\{\{n\}\}$. If $\mathcal{H}_1 = \emptyset$, then $\mathcal{H} \subseteq 2^{[n-1]}$ and hence the result follows by the induction hypothesis. Now suppose $\mathcal{H}_1 = \{\{n\}\}$. Then, since $\mathcal{B}_1 \subseteq \mathcal{H}_1$, we have $\mathcal{C}_1 = \mathcal{C}_2 = \emptyset$, which gives $\mathcal{D} = \mathcal{B}_2 \subseteq \{\emptyset\}$ and $\mathcal{E} = \mathcal{B}_3$. If $\mathcal{D} = \emptyset$, then the argument in Case 1 gives us the result.

Suppose $\mathcal{D} = \{\emptyset\}$. Since $\mathcal{D} = \mathcal{B}_2$, we have $\mathcal{B}_1 = \{\{n\}\}$ and hence $\{n\} \in \mathcal{B}$. By definition of $\mathcal{B}$, $\{1\}$ is also in $\mathcal{B}$. Therefore $\mathcal{B} \neq \mathcal{B}^*$; moreover, since there is no set in $\mathcal{B}\backslash\{\{n\}\}$ intersecting $\{n\}$, $\mathcal{B} = \mathcal{B}'$. Now consider $\mathcal{H}_3$. From $\{1\} \in \mathcal{B} \subseteq \mathcal{H}$ we get $\{1\} \in \mathcal{H}_3$ and hence $\mathcal{H}_3 \neq \{\emptyset\}$. Since $\mathcal{H}$ is hereditary, we have $\emptyset \in \mathcal{H}_3$, meaning that ${\mathcal{H}_3}^* = \emptyset$ and $\mathcal{H}_3 = {\mathcal{H}_3}'$. It follows by the induction hypothesis that $\frac{1}{n}|\mathcal{H}_3| \leq |\mathcal{L}_3|$ (and hence $n|\mathcal{L}_3| - |\mathcal{H}_3| \geq 0$) and that equality holds only if $\mathcal{H}_3 = \{\emptyset\} \cup \binom{[n-1]}{1}$. So we have

$$\begin{aligned}
|\mathcal{L}_3| - \left(|\mathcal{B}^*| + \frac{1}{n+1}|\mathcal{B}'|\right) &= |\mathcal{L}_3| - \frac{1}{n+1}|\mathcal{B}| = |\mathcal{L}_3| - \frac{1}{n+1}(|\mathcal{B}_1| + |\mathcal{B}_3|) \\
&\geq |\mathcal{L}_3| - \frac{1}{n+1}(1 + |\mathcal{H}_3|) = \frac{1}{n+1}((n+1)|\mathcal{L}_3| - 1 - |\mathcal{H}_3|) \\
&= \frac{1}{n+1}(n|\mathcal{L}_3| - |\mathcal{H}_3| + |\mathcal{L}_3| - 1) \geq \frac{1}{n+1}(|\mathcal{L}_3| - 1) \geq 0, \quad (5)
\end{aligned}$$

where the last inequality follows from the fact that $\{1\} \in \mathcal{B} \subseteq \mathcal{H}$ and hence $\{1\} \in \mathcal{L}_3$. So $|\mathcal{B}^*| + \frac{1}{n+1}|\mathcal{B}'| \leq |\mathcal{L}|$. As in Case 1, we have $|\mathcal{A}^*| + \frac{1}{n+1}|\mathcal{A}'| \leq |\mathcal{L}|$ by Lemma 4.1. Suppose equality holds. Then $|\mathcal{B}^*| + \frac{1}{n+1}|\mathcal{B}'| = |\mathcal{L}|$. By the calculation in (5), we must therefore have $\frac{1}{n}|\mathcal{H}_3| = |\mathcal{L}_3|$ $(= 1)$, implying that $\mathcal{H}_3 = \{\emptyset\} \cup \binom{[n-1]}{1}$ (as explained above), and also $|\mathcal{B}_3| = |\mathcal{H}_3|$, implying that $\mathcal{B}_3 = \mathcal{H}_3$. Since $\mathcal{B}_1 = \mathcal{H}_1 = \{\{n\}\}$, $\mathcal{B} = \mathcal{H} = \{\emptyset\} \cup \binom{[n]}{1}$. It clearly follows that $\mathcal{A} = \mathcal{B}$.

Finally, if $\mathcal{A} = \mathcal{H} = \{\emptyset\} \cup \binom{[n]}{1}$, then $\mathcal{A}^* = \emptyset$, $\mathcal{A}' = \mathcal{A}$, $\mathcal{L} = \{\{1\}\}$, and hence $|\mathcal{A}^*| + \frac{1}{n+1}|\mathcal{A}'| = 1 = |\mathcal{L}|$. □

Now for any non-empty family $\mathcal{F}$, let $l(\mathcal{F})$ be the size of a largest intersecting sub-family of $\mathcal{F}$, and let $\beta(\mathcal{F})$ be the largest rational number $c \leq \frac{l(\mathcal{F})}{|\mathcal{F}|}$ such that $|\mathcal{A}^*| + c|\mathcal{A}'| \leq l(\mathcal{F})$ for any sub-family $\mathcal{A}$ of $\mathcal{F}$.

Proof of Theorem 3.11. For any intersecting family $\mathcal{A} \neq \{\emptyset\}$, $\mathcal{A}^* = \mathcal{A}$ and $\mathcal{A}' = \emptyset$. Thus, by Theorem 5.1, $\mathcal{H}\langle x\rangle$ is a largest intersecting sub-family of $\mathcal{H}$ and hence $l(\mathcal{H}) = |\mathcal{S}|$. Since $\emptyset \in \mathcal{H}$, we have $\mathcal{H}^* = \emptyset$ and $\mathcal{H}' = \mathcal{H}$. Thus, by Theorem 5.1 with $\mathcal{A} = \mathcal{H}$, we have $\frac{1}{n+1}|\mathcal{H}| \leq l(\mathcal{H})$ and hence $\frac{1}{n+1} \leq \frac{l(\mathcal{H})}{|\mathcal{H}|}$. Therefore Theorem 5.1 ultimately gives us $\beta(\mathcal{H}) \geq \frac{1}{n+1}$. So we have $k \geq n + 1 \geq \frac{1}{\beta(\mathcal{H})}$.

As in the proof of Theorem 3.8, let $\mathcal{A} = \bigcup_{i=1}^k \mathcal{A}_i$; so $\mathcal{A}^* = \bigcup_{i=1}^k \mathcal{A}_i^*$, $\mathcal{A}' = \bigcup_{i=1}^k \mathcal{A}_i'$ and $|\mathcal{A}'| = \sum_{i=1}^k |\mathcal{A}_i'|$. So we have

$$\sum_{i=1}^k |\mathcal{A}_i| = \sum_{i=1}^k |\mathcal{A}_i'| + \sum_{i=1}^k |\mathcal{A}_i^*| \leq |\mathcal{A}'| + k|\mathcal{A}^*| \leq k\left(|\mathcal{A}^*| + \beta(\mathcal{H})|\mathcal{A}'|\right) \leq kl(\mathcal{H}) = k|\mathcal{S}| \quad (6)$$

and hence, by Lemma 3.4,

$$\prod_{i=1}^k |\mathcal{A}_i| \leq \left(\frac{1}{k}\sum_{i=1}^k |\mathcal{A}_i|\right)^k \leq |\mathcal{S}|^k. \quad (7)$$

We now prove (a) and (b). It is trivial that the conditions in (a) and (b) are sufficient, so it remains to prove that they are also necessary.

Consider first $k > n+1$. Then $k > \frac{1}{\beta(\mathcal{H})}$. From (6) we see that $\sum_{i=1}^k |\mathcal{A}_i| = k|\mathcal{S}|$ only if $|\mathcal{A}'| = 0$ and $|\mathcal{A}_1^*| = \dots = |\mathcal{A}_k^*| = |\mathcal{A}^*| = |\mathcal{S}|$. So $\sum_{i=1}^k |\mathcal{A}_i| = k|\mathcal{S}|$ only if $\mathcal{A}$ is a largest intersecting sub-family of $\mathcal{H}$ and $\mathcal{A}_1 = \dots = \mathcal{A}_k = \mathcal{A}$. It follows from (7) that $\prod_{i=1}^k |\mathcal{A}_i| = |\mathcal{S}|^k$ only if $\mathcal{A}$ is a largest intersecting sub-family of $\mathcal{H}$ and $\mathcal{A}_1 = \dots = \mathcal{A}_k = \mathcal{A}$.

Now consider $k = n+1$. If we still have $k > \frac{1}{\beta(\mathcal{H})}$, then we arrive at the same conclusion as in the previous case $k > n+1$. So suppose $k = \frac{1}{\beta(\mathcal{H})}$. Then $\beta(\mathcal{H}) = \frac{1}{n+1}$.

Suppose $\mathcal{H} \neq \{\emptyset\} \cup \binom{[n]}{1}$. Let $d = \frac{l(\mathcal{H})}{|\mathcal{H}|}$. Since $x \in U(\mathcal{H})$, $\mathcal{S} \neq \emptyset$. Thus, since $|\mathcal{H}| = |\{\emptyset\} \cup \bigcup_{i=1}^n \mathcal{H}\langle i \rangle|$, we get $|\mathcal{H}| \leq 1 + n|\mathcal{S}|$, and equality holds only if $|\mathcal{H}\langle i \rangle| = |\mathcal{S}|$ for all $i \in [n]$. So $|\mathcal{H}| \leq (n+1)|\mathcal{S}|$, and equality holds only if $|\mathcal{S}| = 1$ and $|\mathcal{H}\langle i \rangle| = |\mathcal{S}|$ for all $i \in [n]$. If $i \in [n]$ and $A \in \mathcal{H}\langle i \rangle$, then, since $\mathcal{H}$ is hereditary, all subsets of A containing i are also in $\mathcal{H}\langle i \rangle$; thus, if $|\mathcal{H}\langle i \rangle| = 1$, then $\mathcal{H}\langle i \rangle$ must be $\{i\}$. Therefore, if $|\mathcal{H}| = (n+1)|\mathcal{S}|$, then $\mathcal{H}\langle i \rangle = \{i\}$ for all $i \in [n]$, but this gives the contradiction that $\mathcal{H} = \{\emptyset\} \cup \binom{[n]}{1}$. So $|\mathcal{H}| < (n+1)|\mathcal{S}|$ and hence $\frac{1}{n+1} < \frac{|\mathcal{S}|}{|\mathcal{H}|} = d$. Now let $\mathcal{A} \subseteq \mathcal{H}$. If $\mathcal{A}' = \emptyset$, then obviously $|\mathcal{A}^*| + d|\mathcal{A}'| \leq l(\mathcal{H})$. If $\mathcal{A}' \neq \emptyset$, then $|\mathcal{A}^*| + \frac{1}{n+1}|\mathcal{A}'| < l(\mathcal{H})$ by Theorem 5.1 (as $\mathcal{H} \neq \{\emptyset\} \cup \binom{[n]}{1}$). Thus, if c is the largest rational number such that $c \leq d$ and $|\mathcal{A}^*| + c|\mathcal{A}'| \leq l(\mathcal{H})$ for any $\mathcal{A} \subseteq \mathcal{H}$, then $c > \frac{1}{n+1}$, which is a contradiction since $\beta(\mathcal{H}) = \frac{1}{n+1}$.

We have therefore shown that $\mathcal{H}$ must consist of the sets $\emptyset, \{1\}, \{2\}, \dots, \{n\}$. It follows by the cross-intersection condition that we have the following:

- If one of the families $\mathcal{A}_1, \dots, \mathcal{A}_k$ consists of only one set A and $A \neq \emptyset$, then each of the others either consists of A only or is empty.
- If one of the families $\mathcal{A}_1, \dots, \mathcal{A}_k$ either has more than one set or has the set $\emptyset$, then the others must be empty.

These have the following immediate implications:

- If $\sum_{i=1}^k |\mathcal{A}_i| = k|\mathcal{S}|$, then $\sum_{i=1}^k |\mathcal{A}_i| = n+1$ (since $\mathcal{S} = \{\{x\}\}$ and $k = n+1$) and hence either $\mathcal{A}_1 = \dots = \mathcal{A}_k = \{\{y\}\}$ for some $y \in [n]$, or for some $i \in [k]$, $\mathcal{A}_i = \mathcal{H} = \{\emptyset\} \cup \binom{[n]}{1}$ and $\mathcal{A}_j = \emptyset$ for each $j \in [k] \backslash \{i\}$.
- If $\prod_{i=1}^k |\mathcal{A}_i| = |\mathcal{S}|^k$, then $\prod_{i=1}^k |\mathcal{A}_i| = 1$ and hence $\mathcal{A}_1 = \dots = \mathcal{A}_k = \{\{y\}\}$ for some $y \in [n]$.

Note that for any $y \in [n]$, $\{\{y\}\}$ is a largest intersecting sub-family of $\mathcal{H} = \{\emptyset\} \cup \binom{[n]}{1}$. □

6 Concluding Remarks

As explained in Section 3, the conjectures we suggested in the same section generalise Conjecture 2.1 and hence must be very difficult to prove. However, a problem that arises naturally from our investigation and that should be much more tractable is whether Conjecture 3.5 is true for the case when $\mathcal{H}$ is compressed with respect to an element or at least left-compressed; note that the proof of Proposition 3.6 shows us that it is enough to prove this for the case $k = 2$. This would require a method that is rather different from the one we used because our method is intrinsically designed for the problem of maximising the sum of the sizes (recall that the product part of Theorem 3.11 follows immediately from Lemma 3.4), for which the condition $k \geq n + 1$ is sharp (as shown in Section 3).

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References

[1] I. Anderson, Combinatorics of Finite Sets, Oxford University Press, Oxford, England, 1987.

[2] C. Berge, A theorem related to the Chvátal conjecture, Proceedings of the Fifth British Combinatorial Conference (Univ. Aberdeen, Aberdeen, 1975), pp. 35–40. Congressus Numerantium, No. XV, Utilitas Math., Winnipeg, Man., 1976.

[3] P. Borg, A short proof of a cross-intersection theorem of Hilton, Discrete Math. 309 (2009), 4750–4753.

[4] P. Borg, Cross-intersecting families of permutations, J. Combin. Theory Ser. A 117 (2010), 483–487.

[5] P. Borg, Cross-intersecting families of partial permutations, SIAM J. Discrete Mathematics 24 (2010), 600–608.

[6] P. Borg, I. Leader, Multiple cross-intersecting families of signed sets, J. Combin. Theory Ser. A 117 (2010), 583–588.

[7] P. Borg, On Chvátal's conjecture and a conjecture on families of signed sets, Europ. J. Combin. 32 (2011), 140–145.

[8] P. Borg, The maximum sum and product of sizes of cross-intersecting families, http://arxiv.org/abs/1102.0667.

[9] P. Borg, Extremal t-intersecting sub-families of hereditary families, J. London Math. Soc. 79 (2009), 167–185.

[10] V. Chvátal, http://users.encs.concordia.ca/∼chvatal/conjecture.html.

[11] V. Chvátal, Unsolved Problem No. 7, in: C. Berge, D.K. Ray-Chaudhuri (Eds.), Hypergraph Seminar, Lecture Notes in Mathematics, Vol. 411, Springer, Berlin, 1974.

[12] V. Chvátal, Intersecting families of edges in hypergraphs having the hereditary property, in: C. Berge, D.K. Ray-Chaudhuri (Eds.), Hypergraph Seminar, Lecture Notes in Mathematics, Vol. 411, Springer, Berlin, 1974, pp. 61–66.

[13] P. Erdős, C. Ko, R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford (2) 12 (1961), 313–320.

[14] P. Frankl, The shifting technique in extremal set theory, in: C. Whitehead (Ed.), Combinatorial Surveys, Cambridge Univ. Press, London/New York, 1987, pp. 81–110.

[15] A.J.W. Hilton, An intersection theorem for a collection of families of subsets of a finite set, J. London Math. Soc. (2) 15 (1977), 369–376.

[16] D. Miklós, Some results related to a conjecture of Chvátal, Ph.D. Dissertation, Ohio State University, 1986.

[17] J. Schönheim, Hereditary systems and Chvátal's conjecture, Proceedings of the Fifth British Combinatorial Conference (Univ. Aberdeen, Aberdeen, 1975), pp. 537–539. Congressus Numerantium, No. XV, Utilitas Math., Winnipeg, Man., 1976.

[18] H.S. Snevily, A new result on Chvátal's conjecture, J. Combin. Theory Ser. A 61 (1992), 137–141.

[19] D.L. Wang, P. Wang, Some results about the Chvátal conjecture, Discrete Math. 24 (1978), 95–101.

[20] J. Wang, H. Zhang, Cross-intersecting families and primitivity of symmetric systems, J. Combin. Theory Ser. A 118 (2011), 455–462.

[21] D.B. West, http://www.math.uiuc.edu/∼west/regs/chvatal.html.